Math-109: Pre-Calculus Algebra
Section: 8
Midterm Exam 2 Solutions
Total Points 56

Please write complete step by step solutions (whenever possible) to the problems below.

1. (10 points) Let $f(x) = x^2 + 4x + 3$.

   (a) Write the quadratic function in standard(vertex) form. (Hint: Complete the square or use the alternative method)

   Solution. We know that

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{4}{2} \\ &= -2 \end{aligned}$$

   Hence,

$$\begin{aligned} k &= f(h) \\ &= f(-2) \\ &= (-2)^2 + 4(-2) + 3 \\ &= 4 - 8 + 3 \\ &= -1 \end{aligned}$$

   Thus,

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= 1 \cdot (x - (-2))^2 + (-1) \\ &= (x+2)^2 - 1 \end{aligned}$$

□

   (b) Find the vertex.

   Solution. We have

$$f(x) = (x - (-2))^2 - 1$$

   Hence, comparing to vertex form ($f(x) = a(x-h)^2 + k$) we get, $h = -2$ and $k = -1$.

□

   (c) Find the coordinates of the x-intercepts(zeros).

Solution. We need to solve:

$$\begin{aligned} f(x) &= 0 \\ (x+2)^2 - 1 &= 0 \\ (x+2)^2 &= 1 \\ x+2 &= \pm 1 && \text{Taking square roots on both sides} \\ x &= -2 \pm 1 \\ x = -1 \text{ or } & & x = -3 \end{aligned}$$

Thus, the x-intercepts are $(-1, 0)$ and $(-3, 0)$. □

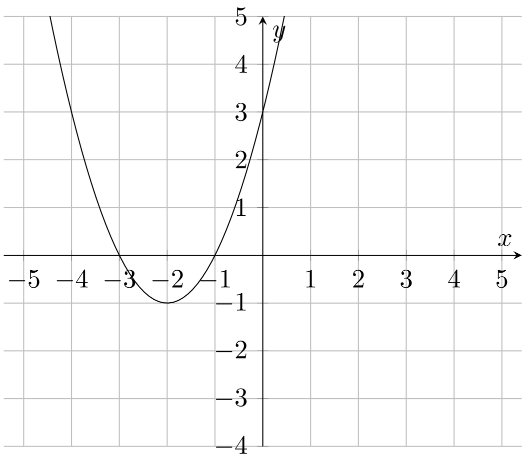
(d) Find the coordinates of the y-intercept.

Solution. We need to find:

$$\begin{aligned} f(0) &= 0^2 + 4 \cdot 0 + 3 \\ &= 3 \end{aligned}$$

Therfore, the y- intercept is $(0, 3)$. □

(e) Sketch the graph and label all the interecepts and vertex.



2. (5 points) Find the quadratic function that has the given vertex and goes through the given point. Write your answer in vertex form.
Vertex: $(3, -5)$, Point:$(5, 7)$.

Solution. Using the vertex form the quadratic function with vertex $(3, -5)$ is given by $f(x) = a(x-3)^2 - 5$. Since the function passes through $(5, 7)$, we have,

$$\begin{aligned} 7 &= a(5-3)^2 - 5 \\ 7 &= a \cdot 4 - 5 \\ 4a &= 12 \\ a &= 3 \end{aligned}$$

Therfore, $f(x) = 3(x-3)^2 - 5$.

□

3. (4 points) Are the following functions polynomials? If they are polynomials, determine their degrees.

   (a) $f(x) = x^7(x-3)^{11}(x+9)$

   Solution. Yes. The degree is 19. □

   (b) $f(x) = x^2 - \frac{1}{x} + 11$

   Solution. No because $\frac{1}{x} = x^{-1}$ which is not allowed for polynomials. □

   (c) $f(x) = x^{10} - x^5 + x^2 + \sqrt[3]{x} + 9$

   Solution. No because $\sqrt[3]{x} = x^{\frac{1}{3}}$ which is not allowed for polynomials. □

   (d) $g(x) = \frac{3(x-3)^2(x-4)(x-9)}{x}$

   Solution. No because we cannot the cancel the x in the denominator and so we get a fraction of polynomials. □

4. (4 points) Consider the polynomial $P(x) = x(x+2)(x-5)^2(x-11)^9$.

   (a) What is the degree of $P(x)$?

   Solution. 1+1+2+9=13 □

   (b) List all the zeros of $P(x)$ including their multiplicites.

   Solution. The zeros are 0(multiplicty 1), -2(multiplicity 1), 5 (multiplicity 2) and 11 (multiplicity 2). □

5. (4 points) Find a polynomial of minimum degree that has the given zeros.

   (a) $-1, \frac{3}{4}, -\frac{1}{6}$

   Solution. $(x+1)(x-\frac{3}{4})(x+\frac{1}{26})$ □

   (b) -10 (multiplicity 3), 7(multiplcity 9)

   Solution. $(x+10)^3(x-7)^9$ □

6. (10 points) Let $f(x) = x^3 - x^2 - 6x$.

   (a) Find the zeros of the polynomial with respective multiplicities.

   Solution. We have

$$\begin{aligned} x^3 - x^2 - 6x &= x(x^2 - x - 6) \\ &= x(x+2)(x-3) \end{aligned}$$

   Thus, the zeros are 0 with multiplicity 1, -2 with multiplicity 1 and 3 with multiplicity 1. □

   (b) Determine whether the graph touches or crosses at each zero(x- intercept).

   Solution. Since all the zeros have odd multiplicities, the graph crosses at all of them. □

   (c) Find the y- intercept.

Solution. We have

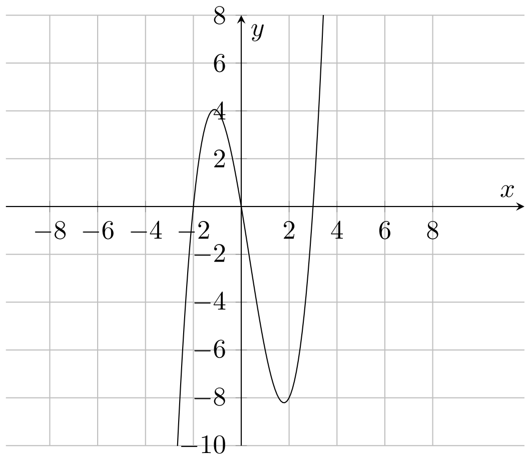
$$\begin{aligned} f(0) &= 0 - 0 - 0 \\ &= 0 \end{aligned}$$

Thus, the y-intercept is $(0, 0)$. □

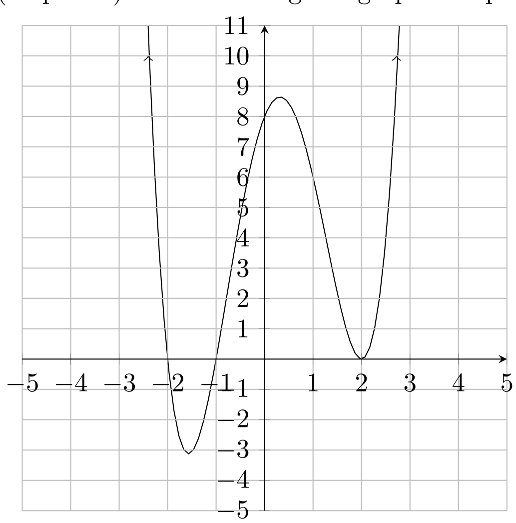
(d) Determine the end behaviors.

Solution. The leading term is x^3 which has odd degree and positive leading coefficient. Thus, as $x \to \infty$, $f(x) \to \infty$, and as $x \to -\infty$, $f(x) \to -\infty$. □

(e) Sketch the graph.



7. (10 points) Consider the given graph of a polynomial.



(a) List each real zero and it smallest possible multiplicity.(Hint: cross vs. touch)

Solution. x=-2 with multiplicty 1 because the graph crosses.
x=-1 with multiplicty 1 because he graph crosses.
x=2 with multiplicty 2 because the graph touches. □

(b) Determine whether the degree of the polynomial is even or odd. (Hint: Remember the 4 cases).

Solution. Comparing the given polynomial to $f(x) = x^2$ we get the degree of the polynomial is even. □

(c) Is the leading coefficient positive or negative?(Hint: Remember the 4 cases)

Solution. The leading coefficient is positive. □

(d) Find the y-intercept.

Solution. (0,8) □

(e) Write the equation for the polynomial. (Assume the least degree possible.)

Solution. $(x+2)(x+1)(x-2)^2$ □

8. (4 points) Divide the following. You can use synthetic division for (b).

(a) $x^3 + 6x^2 - 2x - 5 \div x^2 - 1$

Solution.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\ \ x+6 \\ \hline x^2-1 & x^3 + 6x^2 - 2x - 5 \\ & -x^3 \quad\quad\quad\ \ + x \\ \cline{2-2} & \quad\quad 6x^2 \ \ - x - 5 \\ & \quad -6x^2 \quad\quad\ \ + 6 \\ \cline{2-2} & \quad\quad\quad\quad\ \ -x + 1 \end{array}$$

□

(b) $x^4 - x^3 - 2x + 2 \div x + 1.$

Solution.
$$\begin{array}{r|rrrrr} & 1 & -1 & 0 & -2 & 2 \\ -1 & & -1 & 2 & -2 & 4 \\ \hline & 1 & -2 & 2 & -4 & 6 \end{array}$$

Thus, $\frac{x^4-x^3-2x+2}{x+1} = x^3 - 2x^2 + 2x - 4 + \frac{6}{x+1}$ □

9. (5 points) Let $P(x) = x^5 - 2x^4 + x^3 - 2x^2 - 2x + 4$.

(a) Use Descartes' rule of signs to determine the possible number of positive zeros for $P(x)$.

Solution. The number of sign changes for $P(x)$ is 4. Thus, there are either 4, 2 or 0 positive zeros. □

(b) Use Descarets' rule of signs to determine the possible number of negative zeros for $P(x)$. (Note that you have to find $P(-x)$.)

Solution. Note that $P(-x) = -x^5 - 2x^4 - x^3 - 2x^2 + 2x + 4$. The number of sign changes for $P(-x)$ is 1. Thus, there is 1 negative zero.

□

(c) Use the rational zero test to determine the possible rational zeros.

Solution. The possible rational zeros are of the form $\frac{\text{factor of} a_0}{\text{factor of } a_n}$. Since $a_0 = 4$ and $a_n = 1$, the possible rational zeros are $\frac{\{\pm1,\pm2,\pm4\}}{\{\pm1\}}$. Thus, the possible rational zeros are $\pm1, \pm2, \pm4$. □

(d) Test which ones are the zeros and factor the polynomial as a product of linear and/or irreducible quadratic factors. (Note that you don't need to test for each and every zero. Make use of your answer to (a) and (b).)

Solution. Let us check whether 1 is a zero or not. This is the same as checking whether $x-1$ is a factor or not. Hence, using synthetic division we have

$$\begin{array}{c|cccccc} & 1 & -2 & 1 & -2 & -2 & 4 \\ 1 & & 1 & -1 & 0 & -2 & -4 \\ \hline & 1 & -1 & 0 & -2 & -4 & 0 \end{array}$$

Hence, 1 is a zero. Now to check whether 2 is a zero or not we will divide the quotient by $x-2$. We have

$$\begin{array}{c|ccccc} & 1 & -1 & 0 & -2 & -4 \\ 2 & & 2 & 2 & 4 & 4 \\ \hline & 1 & 1 & 2 & 2 & 0 \end{array}$$

Hence, 2 is a zero. Note that there cannot be 4 zeros. So we need not check for 4. Now let us look for negative zeros.
Let us check whether -1 is a zero or not. This is the same as checking whether $x+1$ is a factor or not. Hence, using synthetic division we have

$$\begin{array}{c|cccc} & 1 & 1 & 2 & 2 \\ -1 & & -1 & 0 & -2 \\ \hline & 1 & 0 & 2 & 0 \end{array}$$

Thus, $P(x) = (x-1)(x-2)(x+1)(x^2+2)$.

$\square$